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Quantum deletion is possible

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Abstract

A deleting operation is introduced which differs from the commonly used *controlled-not* (C-not) conditional logical operation —to flip the (classical or quantum) state of the last copy in a chain in a deletion process. It is completely reversible, in the classical case, possessing a most natural cloning operation counterpart. We call this deleting procedure R-deletion since, in a way, it can be viewed as a 'randomization' of the standard C-not operator. It is a nonlinear operation and has the remarkable property of avoiding in a simple manner the 'impossibility of deletion of a quantum state' principle, put forward by Pati and Braunstein recently [1].

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The possiblity of constructing a consistent quantum information theory and, what is more, of making practical use of its impresive potentialities (e.g., quantum cryptography [2], quantum teleportation [3, 4], quantum computing [5], etc.) has attracted considerable attention in the Physics community during the last twenty years [5]. Almost as old is, however, the quantum 'no-cloning theorem', due to Wooters and Zurek [6], and Dieks [7], which prevents the replication of unknown quantum states. In spite of this result −which does not allow, in particular, quantum information to be amplified accurately− and as it usually happens with no-go theorems, different alternatives to circumvent this strict prohibition have appeared in the literature [8]-[13]. They restablish, in several different ways, a sort of consistency between the quantum theory and its possible real application as an information theory, which will obviously have the processes of copying, storing, and retrieving of information as some of its most basic tasks. However, such consistency has been regained, at the very best, only at the level of arbitrarily good approximations in specific circumstances, while the no-cloning theorem still remains as a cornerstone in this field.

Recently, Pati and Braunstein have formulated what looks a complement to this theorem, namely a proposition that can be termed as a quantum 'no-deleting theorem' [1]. In the same way as the no-cloning theorem told us that, contrary to what happens in classical information issues, the cloning of quantum information cannot be taken for granted, it also would now happen that deleting quantum information is in no way a trivial operation. One might argue, at first sight, that this result sounds quite old and well known (see Szilard [14] and Landauer [15]). In fact, replacing a sequence of 0's and 1's by a perfectly ordered state of 0's is thermodynamically costly, at the classical level. But the concept of deletion used in [1] has a more subtle formulation, which prevents the result from being trivial, and makes it much resemble the inverse of the usual cloning operation in [6, 7], in the sense that it consists in the deletion of just one copy of a bit that is kept in several (at least two) copies. In other words, this sort of deletion can only start when one has two arbitrary, unknown but identical, bits (which can be termed 'original' and 'copy').

To be precise, and for the benefit of the reader, let us here recall the controlled not (or C-not) operation of classical information theory, which allows the deletion and copying of an arbitrary sequence of classical bits (see, for instance, Zurek [16]). Confronted with a pair of states, $|s_1> |s_2>$, C-not yields:

$$\begin{aligned} &|0>|0> \longrightarrow |0>|0>, \quad |0>|1> \longrightarrow |0>|1>, \\ &|1>|0> \longrightarrow |1>|1>, \quad |1>|1> \longrightarrow |1>|0>, \end{aligned} \tag{1}$$

that is, it replaces the second bit by its opposite whenever the first bit is $|1>$, and leaves it untouched, when the first is $|0>$. It is immediate that, classically, this operation clones the first component of a sequence of pairs of states (by imprinting it in the second component), when acting on a sequence of pairs where the second components are all 0's (blank sequence). Thus, for instance,

$$
\begin{aligned}
&|0>|0>, |0>|0>, |1>|0>, |0>|0>, |1>|0>, |1>|0>, \ldots \\
&\qquad \longrightarrow |0>|0>, |0>|0>, |1>|1>, |0>|0>, |1>|1>, |1>|1>, \ldots
\end{aligned} \tag{2}
$$

Moreover, this operation is reversible: when confronted with a sequence of identical pairs of states (such as for instance the second one in (2)), the C-not operation will turn out a sequence where the first components remain unchanged, while the second ones are all 0's. It is rather immediate (see e.g. [16]) that this simple operation fails altogether as a copying or deleting tool when applied to pairs of *quantum* bits (qubits). In terms of the states $|0>$ and $|1>$ a qubit will have the general form $|s>= \alpha|0> + \beta|1>$ (α and β being complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$) and, the C-not operation acting, for instance, on the pair $|s>|0>$ yields an entangled state:

$$
(\alpha|0> + \beta|1>)|0> \longrightarrow \alpha|0>|0> + \beta|1>|1> \tag{3}
$$

(the C-not operation is linear). Put it in this way, it looks like there is no way out of the conclusion in the paper by Pati and Braunstein [1] $-$to which the reader is addressed for conventions, notation and additional references$-$ that it is impossible to delete a copy of a quantum state, that comes in at least two copies, $|s>|s>$. Our main idea here will be to use a different form of the C-not operation together with an alternative notion of deletion, which depart from but generalize in a way the definitions above.

We shall now fix our strategy. We give the term *deletion* the same sense of Ref. [1], as has been just described and which, in the words of Zurek [16], may be termed a narrow concept, but one that is being indeed most widely employed. This means, in what follows we shall just (as above) delete a single copy of some classical or quantum information piece that is kept in some device in at least two copies, so that at least one copy of the information remains in the end. But, as already advanced, our logical deleting/cloning operations will be *different* from the C-not operation. They will be called, respectively, *random deletion* and *cloning from random*, the first being a genuine R-choice. As it happens with the C-not operation, they also can operate both on classical and quantum systems, without problem. Being more specific, classical R-deletion is defined as:

$$
\left\{
\begin{array}{l}
|0> \longrightarrow R|0> = \left\{ \begin{array}{ll} |0>, & \text{with} \;\; p = 1/2, \\ |1>, & \text{with} \;\; q = 1-p = 1/2, \end{array} \right. \\
|1> \longrightarrow R|1> = \left\{ \begin{array}{ll} |0>, & \text{with} \;\; p = 1/2, \\ |1>, & \text{with} \;\; q = 1-p = 1/2, \end{array} \right.
\end{array}
\right. \tag{4}
$$

where p and q denote probabilities and we have obviated the extra copies. That is, any state of the classical system is replaced by a reference state, which is not determined to be the $|0, 0, 0, \ldots, 0>$ state, but rather one chosen completely at random. Plainly, we would, for instance, delete the information contained in a Shakesperian play not by replacing all the characters with say a's, but by replacing it with say[2] a similar play written by a monkey sitting in front of a typewriter (that is, an arbitrary state in a thermal bath, rather than a chosen, standard, zero-state [17, 18]). Of course, it will not always be the same play and the obvious question could be asked: how can we know *a priori*, on looking to a track or a whole disk, that it has undergone a deletion process, e.g., that it is empty and does not contain any information, if spins are not all alligned as in the usual $|0, 0, 0, \ldots, 0>$ state? One of the possible answers: we know that the device is 'empty' because it is labeled as such. For this we will just need an additional bit. To give a further visual picture of this last issue, let us mention that it freely corresponds to what we do at home when we decide that a video-tape or a CD are ready for re-use, after we have seen the movie (or are just fed up with the music) that we had previously recorded on them. In short, the new bit just mentioned should mimic this practice. Note, moreover, that what one never does in fact is to physically erase a recorded tape before recording it again (well, this was actually performed in the very old times!).

R-cloning has no essential difference with respect to the ordinary logical operation 'inverse' to C-not in Eq. (1) [6, 7] (see also the very clear description by Zurek [16]). By presenting pairwise the state to be cloned together with the random state (whatever it may be), R-cloning will replace the last with the first. There is nothing really new with the procedure in this case.

In complete analogy with (4), quantum R-deletion starts by considering first an infinite family of quantum states, labeled by a random parameter. Let us take, for instance, the family of one-dimensional wave functions $\varphi_\sigma(x)$ given by Gaussian distributions N(0, σ), with σ a random positive real number that labels the corresponding family of quantum states $|\varphi_\sigma>$. Using the same notation of Pati and Braunstein [1], let us consider a couple of identical qubits (as two photons of arbitrary polarization) in some quantum state $|\psi>$ together with an ancilla in a state $|A>$, corresponding to the 'ready' state of the deleting device [1]. The aim of the deleting device in the spirit of Ref. [1] was to replace one of the two copies of $|\psi>$ with some standard, fixed state of a qubit $|\Sigma>$. However, in the spirit of our approach to deletion (as extensively described above, in the classical case), our

[2] I apologize for the example

R-deletion operator yields, in the quantum situation:

$$|\psi>|\psi>|A>\longrightarrow|\psi>|\Sigma_{\sigma_1}>|A_\psi>, \tag{5}$$

where now the standard state of a qubit ($|\Sigma>$ in [1]) is replaced with one of the infinite family, chosen at random ($|\Sigma_{\sigma_1}>$ here).

When we now consider the action of R-deletion on a pair of horizontally and vertically polarized photons, respectively, we obtain

$$|H>|H>|A> \quad\longrightarrow\quad |H>|\Sigma_{\sigma_2}>|A_H>, \tag{6}$$

$$|V>|V>|A> \quad\longrightarrow\quad |V>|\Sigma_{\sigma_3}>|A_V>, \tag{7}$$

and all odds are against having the same standard state for the deleted copy, namely $|\Sigma_\sigma>$ (therefore the three different labels). By applying now R-deletion to an arbitrary input qubit $|\psi>=\alpha|H>+\beta|V>$ (with $|H>$ and $|V>$ forming a basis, and α and β being complex numbers such that $|\alpha|^2+|\beta|^2=1$), we obtain

$$|\psi>|\psi>|A>\longrightarrow\alpha^2|H>|\Sigma_{\sigma_2}>|A_H>+\beta^2|V>|\Sigma_{\sigma_3}>|A_V>+\sqrt{2}\,\alpha\beta|\Phi>, \tag{8}$$

where $|\Phi>$ is the state obtained by R-deletion of the entangled state $(1/\sqrt{2})(|H>|V>+|V>|H>)|A>$. But, whatever this state be in our actual realization of R-deletion, we do *not* recover now, from the linearity of Quantum Mechanics, the result in [1] that $|A_\psi>=\alpha|A_H>+\beta|A_V>$. In fact, the linearity is *lost* in the R-deletion process (the most obvious reason being, that there is now no standard $|\Sigma>$ state to play with). In other words, R-deletion is *not* a linear process.

A more precise argument, to be compared with the one in Ref. [1], goes as follows. R-deletion ceases to be a linear process simply because it does not satisfy the textbook definition of linearity [19]. From a different viewpoint, if it were linear, we immediately would arrive to a contradiction. For the same reason, neither is the linearity of the relation $|\psi>=\alpha|H>+\beta|V>$ transfered to the ancilla state $|A_\psi>$, what would eventually keep the whole information and prevent its deletion (as was argued in [1]). In fact, R-deletion acting on $|\psi>|\psi>|A>$ yields (5) which, on the other hand, if $|A_\psi>$ were to be given by $|A_\psi>=\alpha|A_H>+\beta|A_V>$, by taking into account (6) and (7), would yield the identity:

$$\begin{aligned}\alpha^2|H>|\Sigma_{\sigma_2}>|A_H>+\beta^2|V>|\Sigma_{\sigma_3}>|A_V>+\sqrt{2}\alpha\beta|\Phi>\\ =\alpha^2|H>|\Sigma_{\sigma_1}>|A_H>+\beta^2|V>|\Sigma_{\sigma_1}>|A_V>\\ +\alpha\beta\left(|H>|\Sigma_{\sigma_1}>|A_V>+|V>|\Sigma_{\sigma_1}>|A_H>\right),\end{aligned} \tag{9}$$

which is impossible to fulfill given the nature of R-deletion (in particular, the randomness of the σ_i, $i=1,2,3$). Given that the states $|\Sigma_{\sigma_i}>$, $i=1,2,3$, are completely unrelated, it is now impossible to recover information about the state $|\psi>$ from that of the ancilla remnant $|A_\psi>$. In conclusion, we have actually managed to delete one copy of information, in the end.

R-deletion does not seem to have much new to say on results about cloning. Notwithstandingthat, if one considers cloning as (in some sense) an inverse procedure to deleting, we are here in the presence of two sensibly different deleting operations to which the same cloning operation appears to be a common inverse, in the classical case. This is in fact not completely true. Strictly speaking, the cloning operation which is the inverse of R-deletion actually contains the cloning operation that is the inverse of C-not. This is so, since the last cloning will always act on the standard blank state $|0,0,0,\ldots,0>$ or $|\Sigma>$ state only (as the second state, to be cloned). Only provided its definition were extended, allowing it to act on *any* state as second in the pair, then it would coincide with the cloning which is the (classical) inverse of R-deletion.

To summarize, the new logical operation, R-deletion, that has been here introduced, is *not* a linear process and thus avoids the linear transference of the information kept in any arbitrary quantum state to the ancilla, as it disastrously happened with the ordinary deletion procedure [1], what prevented even the least amount of deletion of quantum information in that case.

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